Problem 1 (25 points)

(from Liou, 1.23) Consider an isothermal non-scattering atmosphere with a temperature T and let the surface temperature of such atmosphere be Ts. Derive an expression for the emergent flux density at the top of atmosphere whose optical depth is τ^* by using Eq. (1.4.25) and show that is can be expressed by the exponential integral of third order given by

$$E_3(\tau^*) = \int_0^1 \exp(-\tau^*/\mu)\mu d\mu$$

Solution:

The upward intensity at the top of atmosphere is given by Eq. (1.4.25) in the text in the form

$$I(0;\mu;\phi) = I(\tau_*;\mu;\phi)\exp(-\tau_1/\mu) + \int_0^{\tau_*} J(\tau';\mu;\phi)\exp(-\tau'/\mu)\frac{d\tau'}{\mu}$$

Since

$I(\tau_*;\mu;\phi) = B_\nu(T_s)$ (surface emission)

$J(\tau;\mu;\phi) = B_\nu(T)$ (atmospheric emission),

we find

$$I(0;\mu) = B_\nu(T_s)\exp(-\tau_*/\mu) + \int_0^{\tau_*} B_\nu(T)\exp(-\tau'/\mu)\frac{d\tau'}{\mu} =$$

$$= B_\nu(T_s)\exp(-\tau_*/\mu) + B_\nu(T)\left[1-\exp(-\tau_*/\mu)\right]$$

The flux density is defined by

$$F(0) = \int_0^{2\pi}\int_0^1 I(0,\mu)\mu d\mu d\phi = 2\pi\int_0^1 I(0,\mu)\mu d\mu =$$

$$= \pi B_\nu(T_s)2\int_0^1 \exp(-\tau_*/\mu)\mu d\mu + \pi B_\nu(T)2\int_0^1 \left[1-\exp(-\tau_*/\mu)\right]\mu d\mu =$$

$$= \pi B_\nu(T_s)2E_3(\tau_*) + \pi B_\nu(T)\left[1-2E_3(\tau_*)\right]$$

where

$$E_3(\tau^*) = \int_0^1 \exp(-\tau^*/\mu)\mu d\mu$$

is the exponential integral of the third order.

Problem 2 (20 points).

The scale height H is defined by dp/p =-dz/H. From the hydrostatic equation and the equation of state, show that H=kT/Mg. where K is the Boltzman constant, M is the mass of air molecules, and g is gravity. Since the molecular transitional energy is 0.5KT, the scale height is then twice the distance through which atoms/molecule that have the equipartition of translation energy can rise in the vertical direction against the force of gravity. For T=296 and T=50 calculate atmospheric scale height H.

Solution.

The scale height, H, is defined by

$$\frac{dp}{p} = -\frac{dz}{H}$$

The hydrostatic equation is given by

$$\frac{1}{\rho}\frac{dp}{dz} + g = 0.$$

Substituting the equation of state p=ρRT, where R is the gas constant for air, yields

$$\frac{dp}{p} = -\frac{dzg}{R_g T}.$$

The scale height is then given by

$$H = \frac{R_g T}{g} = \frac{KT}{Mg},$$

where $K=M*R_g=(\mu/N_a)*(R/\mu)= R/N_a$ is the Boltzman constant, R is the universal gas constant (8.31432 J $mole^{-1}$ K^{-1}), M is mass of air molecule (g), μ is the molecular weight for air (29 g mol^{-1}), N_a is Avogadro number (6.02297x 10^{23} molecule mol^{-1}), and g (9.8 m/sec) represents the gravitational attraction force. The translation energy is given by KT/2. It follows that the scale height is twice the distance through which atoms or molecules (having the equipartition of translation energy) can rise in the vertical direction against of force of gravity. The atmospheric scale height is 8 km for T 296 K and about 2 km at 50 K.

Problem 3 (50 points)

1. *Use the approximation for the Ladenburg and Reiche function in the notes to calculate the equivalent width of the 183.3 GHz water vapor line for two layers. The first layer (0 to 1 km in a midlatitude summer atmosphere) has* $u = 1.15\ \mathrm{g/cm^2}$, *pressure* $p = 958\ \mathrm{mb}$, *and temperature* $T = 292\ \mathrm{K}$. *For the first layer the line halfwidth is* $\alpha = 0.0912\ \mathrm{cm^{-1}}$ *and line strength is* $S = 2.66\ \mathrm{cm/g}$. *The second layer (12 to 13 km in a midlatitude summer atmosphere) has* $u = 0.000392\ \mathrm{g/cm^2}$, *pressure* $p = 194\ \mathrm{mb}$, *and temperature* $T = 219\ \mathrm{K}$. *For the second layer the line strength is* $S = 4.29\ \mathrm{cm/g}$. *Calculate the line halfwidth, given that the halfwidth temperature coefficient is 0.64.*

   What curve of growth regime (limit) is each layer in? How close are the equivalent width formulas for these limits?

Solution

The equivalent width of a single absorption line is

$$W = 2\pi\alpha L(x) \qquad x = \frac{Su}{2\pi\alpha}$$

where α is the line halfwidth, S is the line strength, and u is the absorber amount. The Ladenburg and Reiche function can be approximated with a maximum error of 1% near $x = 1$ by

$$L(x) = x[1 + (\pi x/2)^{5/4}]^{-2/5}$$

For the 0-1 km layer the parameter x is

$$x = \frac{(2.66 \text{ cm/g})(1.15 \text{ g/cm}^2)}{2\pi(0.0912 \text{ cm}^{-1}} = 5.34$$

The Ladenburg and Reiche approximation and equivalent width are

$$L(x) = 1.794 \quad W = 1.028 \text{ cm}^{-1}$$

The actual Ladenburg and Reiche function is 1.814 or $W = 1.040 \text{ cm}^{-1}$.

For the 12-13 km layer we first have to find the absorption line halfwidth by scaling the halfwidth from the lower layer:

$$\alpha = \alpha_0 \left(\frac{p}{p_0}\right)\left(\frac{T_0}{T}\right)^n = (0.0912 \text{ cm}^{-1})\left(\frac{194}{958}\right)\left(\frac{292}{219}\right)^{0.64} = 0.0222 \text{ cm}^{-1}$$

The x parameter is

$$x = \frac{(4.29 \text{ cm/g})(0.000392 \text{ g/cm}^2)}{2\pi(0.0222 \text{ cm}^{-1}} = 0.012$$

and the Ladenburg and Reiche function and equivalent width are

$$L(x) = 0.012 \quad W = 0.00168 \text{ cm}^{-1}$$

The equivalent width is much smaller due to the very low amount of water vapor in the upper troposphere.

The curve of growth regime can be determined from x parameter. The line center optical depth is $\tau_{cen} = 2x$. For the 0-1 km layer the center optical depth is greater than 10, and so the line is clearly saturated. This is the strong line limit. The equivalent width in this limit is proportional to $\sqrt{u}$:

$$W_{strong} = 2\sqrt{Su\alpha} = 2\sqrt{(2.66 \text{ cm/g})(1.15 \text{ g/cm}^2)(0.0912 \text{ cm}^{-1})} = 1.056 \text{ cm}^{-1}$$

which is close to the actual value above.

For the 12-13 km layer $x = 0.012$ so this is the weak line limit where the equivalent width is linear in the absorber amount:

$$W_{weak} = Su = (4.29 \text{ cm/g})(0.000392 \text{ g/cm}^2) = 0.00168 \text{ cm}^{-1}$$

2. *Show the sensitivity of band mean transmission to pressure by plotting the Goody random band model transmission as a function of pressure. Use the 400 to 500 cm^{-1} portion of the pure rotational water vapor band, for which the band model parameters at 260 K and 1013 mb are* $\bar{S}/\delta = 9.0\,\mathrm{m^2/kg}$ *and* $\bar{S}/\bar{\alpha}\pi = 103\,\mathrm{m^2/kg}$.

   a) Graph the Goody band model transmission as a function of pressure (log scale for p from 1 to 1000 mb) for water vapor absorber amount of $u = 20\,\mathrm{kg/m^2}$ (2 cm). Assume the temperature is fixed at $T = 260$ K.

Solution

The Goody random band model transmission is

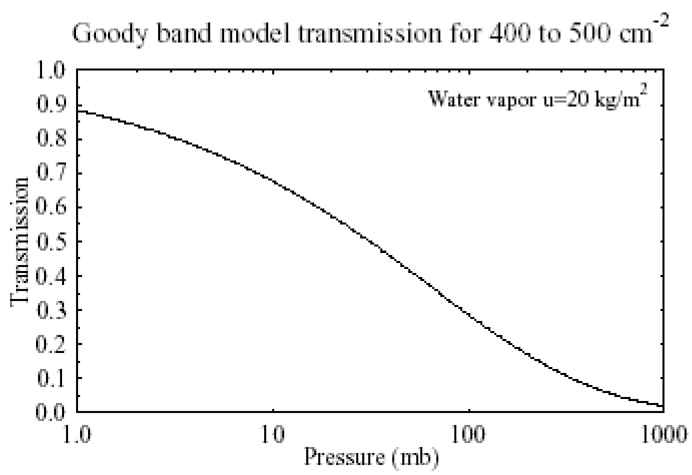
$$\mathcal{T}(u) = \exp\left[-\frac{\bar{S}u}{\delta}\left(1+\frac{\bar{S}u}{\pi\bar{\alpha}}\right)^{-1/2}\right]$$

where $\bar{S}$ is the mean line strength and $\bar{\alpha}$ is the mean line width. The mean line width is proportional to pressure

$$\bar{\alpha} = \bar{\alpha}_0\left(\frac{p}{p_0}\right)$$

where $\bar{\alpha}_0$ is the mean line width at the reference pressure $p_0 = 1013$ mb.

The plot of band mean transmission vs. pressure shows that the transmission falls from 0.88 at 1 mb to 0.02 at 1000 mb.



b) Explain the change in transmission with pressure in terms of absorption line physics.

Solution

For fixed temperature the absorption line strengths do not change, and thus the mean optical depth is constant because the absorber amount is fixed. The effect of increasing pressure is to increase the width of the absorption lines. Many of the water vapor lines are strong so that the line cores remain saturated ($\tau_\nu \gg 1$) as the pressure is increased. Therefore, the effect of the increasing line width of these lines is to increase the fraction of the spectrum with essentially zero transmission. Another way of putting this is to say that the equivalent width of the lines increases with pressure in the strong line limit.

Extra Problem 4 (30 points).

From (Liou, 4.6) The half-width of Lorentz line is proportional to pressure and can be expressed by $\alpha \approx \alpha_r (p/p_r)$, where α_r is the half width at the reference pressure p_r. Show that the optical depth may be expressed by

$$T_\nu = \exp(-\tau) = \left[\left(\nu^2 + \alpha_1^2\right) \Big/ \left(\nu^2 + \alpha_2^2\right) \right]^\lambda$$

where α_1 and α_2 are two integration limits and

$$\lambda = Sp_r q \big/ \left(2\pi g \alpha_r\right)$$

where q is the mixing ratio and g is the gravitational acceleration.

Solution

The optical depth of a Lorentz line may be expressed by

$$\tau = \int k_\nu du \, .$$

The half width of a Lorentz line may be expressed by

$$\alpha \cong \alpha_r \left(\frac{p}{p_r} \right).$$

Thus, absorption coefficient for a Lorentz line can be written as

$$k_\nu = \frac{S}{\pi} \frac{\alpha}{\nu^2 + \alpha^2} = \frac{S}{\pi} \frac{\alpha_r (p/p_r)}{\nu^2 + \alpha_r^2 (p/p_r)^2} \, .$$

The path length is related to pressure via the hydrostatic equation:

$$du = -\rho_a dz = -\rho_a \cdot (-dp/\rho g) = q \, dp/g \, ,$$

where the mixing ratio $q=\rho_a/\rho$, with ρ_a and ρ being the densities fro the absorbing gas and air respectively. It follows that

$$\tau = \int_{p1}^{p2} \frac{S}{\pi} \frac{\alpha_r (p/p_r)}{\nu^2 + \alpha_r^2 (p/p_r)^2} \cdot \frac{q}{g} dp =$$

$$= \frac{S}{2\pi} \frac{q}{g} \frac{p_r}{\alpha_r} \cdot \int_{p1}^{p2} \frac{d\left(\alpha_r^2 (p/p_r)^2\right)}{\nu^2 + \alpha_r^2 (p/p_r)^2} =$$

$$= \frac{S}{2\pi} \frac{q}{g} \frac{p_r}{\alpha_r} \cdot \ln\left(\nu^2 + \alpha_r^2 (p/p_r)^2\right) \Big|_{p1}^{p2} =$$

$$= \lambda \ln \frac{\nu^2 + \alpha_1^2}{\nu^2 + \alpha_2^2} = \ln \left(\frac{\nu^2 + \alpha_1^2}{\nu^2 + \alpha_2^2} \right)^\lambda$$

where

$$\lambda = \frac{S}{2\pi} \frac{q}{g} \frac{p_r}{\alpha_r}, \quad \alpha_1 = \alpha_r \frac{p_1}{p_r}, \quad \alpha_2 = \alpha_r \frac{p_2}{p_r} \, .$$

Thus, we have

$$\exp(-\tau) = \exp\left(-\ln\left(\frac{\nu^2 + \alpha_1^{\ 2}}{\nu^2 + \alpha_2^{\ 2}}\right)^{\lambda}\right) = \left(\frac{\nu^2 + \alpha_1^{\ 2}}{\nu^2 + \alpha_2^{\ 2}}\right)^{\lambda}$$

Extra Problem 5 (30 points).

a) Why does the 10 cm^{-1} resolution atmospheric transmission increase going from the band center to the band edge of an absorption feature? Explain in terms of the physics of molecular absorption. Use the 15 μm CO_2 band as an example.

Solution:

The transmission from space to a given level increases as the wave-number moves away from the vibrational band center. This implies the lines toward the band edges tend to have a smaller optical depth and thus lower mass absorption coefficient. The absorption coefficient

$$k_{m,\nu} = \frac{S\alpha_L/\pi}{(\nu-\nu_0)^2 + \alpha_L^2}$$

is higher in the band center because the line strength S is larger there. The line half-width α_L is mainly affected by pressure and the average pressure does not change with the wave-number. The line strength decreases in the band edges, because those lines are from rotational transitions in high energy levels (large J). These large J rotational energy levels have few molecules in them because their energy is significantly above typical thermal energies in the atmosphere.

b) The 708 to 724 cm-1 transmission from space to 10 km is 0.765, while from 5 to 4 km it is only 0.425, even though these two paths contain virtually the same absorber amount u of CO2 (Δp=76 mb in each case). Explain this difference in transmission in terms of the behavior of molecular absorption lines.

Solution:

Here we have a particular wave-number range, so we are considering the same absorption lines. However, the different height corresponds to average pressures of about 40 *mb* (for space to 18 km) and 580 *mb* (for 5 to 4 km). The 15 times greater pressure causes the lines widths t be 15 times wider. Although the higher pressure reduces the optical depth of the lines centers, they generally remain large, so the transmission of the line center remains zero (T=exp($-\tau_\nu$)). The higher pressure causes the optical depth of the wings to increase, and this effect reduces the band mean transmission.